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The non-Gorenstein locus of toric varieties

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Abstract. We describe the non-Gorenstein loci of normal toric varieties. In the case of Hibi rings a complete combinatorial description of this locus is provided in terms of the underlying poset. As a non-toric application we compute the dimensions of the non-Gorenstein loci of the first secant variety of Segre varieties.

Zusammenfassung Wir bestimmen den nicht-Gorenstein Lokus normaler torischer Varietäten. Für Hibi-Ringe wird eine vollständige kombinatorische Charakterisierung mithilfe der zugrundeliegenden partiell geordneten Menge gegeben. Zuletzt wird als eine nicht-torische Anwendung die Dimension des nicht-Gorenstein Lokus der ersten Sekantenvarietät von Segre Varietäten berechnet.

Eidesstattliche Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und eigenhändig sowie ohne unerlaubte fremde Hilfe und ausschließlich unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe.

Datum, Ort

Unterschrift

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1 Introduction

Combinatorial algebraic geometry constitutes the part of algebraic geometry whose objects can be described in terms of combinatorial data such as matroids, polytopes, partially ordered sets and fans. The fruitful interplay between combinatorics and geometry has many manifestations. Geometric phenomena such as degenerations of varieties, their intersections and their singularities can be related to combinatorics using Newton-Okounkov bodies, or other methods from tropical and toric geometry. On the other hand profound results about Polytopes can be extracted from the associated toric varieties by applying algebro-geometric methods. Another example of methods from algebraic geometry inspiring results in combinatorics is the proof of log-concavity of matroid h-vectors by Huh and Katz [HK12].

In this thesis we study toric varieties and focus on a local measure for singularity which is based on a natural generalization of smoothness, the Gorenstein property. Particular attention will be given to a class of toric rings called Hibi rings. Their strong combinatorial flavor comes from an underlying finite partially ordered set whose properties reflect in the geometry of the associated toric variety.

1.1 Motivation

There is a multitude of regularity conditions on varieties of different restrictiveness, the most prominent being smoothness. A variety is smooth exactly at those places where the top exterior power of the cotangent sheaf can be formed, the canonical line bundle. This object plays an important role in Serre-duality and more general duality statements. Other regularity conditions can be rephrased in terms of the existence of certain dualizing objects as well. In particular if a proper variety is Cohen-Macaulay then there exists a dualizing sheaf ω satisfying Serre duality and it is Gorenstein if in addition ω is a line-bundle, naturally generalizing smoothness. In this thesis we measure how far a variety deviates from the Gorenstein property by computing the subvariety Z formed by all places where it fails to hold, consisting of those points x in X for which the stalk $\mathcal{O}_{X,x}$ is not a Gorenstein ring.

For toric varieties there are combinatorial criteria for the Gorenstein property which we will generalize in order to completely characterize Z. Special attention will be given to a class of toric rings for which the non-Gorenstein locus has already been investigated in several papers. A Hibi ring $k[P]$ is a toric algebra associated to a finite partially ordered set P whose combinatorial properties determine the geometry of the associated toric variety $\mathrm{Spec}(k[P])$. Hibi rings were introduced 1987 by Takayuki Hibi in [Hib87]. They were studied initially because of their appearance as flat degenerations of the coordinate rings of Grassmannians and, more generally, flag varieties. For a construction of these deformations using Sagbi bases consider [CHT06] and for more information on Hibi rings and their relations to algebras with straightening laws see [CEP82].

There are several known results that relate the locus Z to the combinatorics of P. Already in [Hib87] it was recognized that X is Gorenstein if and only if P is pure. In [HMP19] further results in this direction are shown: Z is zero-dimensional if and only if each connected component of P is pure. Other papers investigate a natural non-reduced structure of Z. In fact, for every affine Cohen Macaulay variety X there is a canonical module ω over the coordinate ring R generalizing the canonical bundle Ω_X^n from the smooth case and Z can be shown to be the reduced vanishing locus of the trace ideal

$$\operatorname{tr}(\omega) = \sum_{\varphi \in \operatorname{Hom}_R(\omega, R)} \varphi(\omega).$$

In an attempt to find good regularity conditions that are stronger than the Cohen Macaulay property and weaker than the Gorenstein property, the non-reduced structure of Z given by $\operatorname{tr}(\omega)$ is studied in [HHS19]. For positively graded R (for example R a toric algera) X is defined to be nearly Gorenstein if $\operatorname{tr}(\omega)$ contains the unique maximal graded ideal. This is a stronger condition than Z being of dimension zero and is shown to hold for a Hibi ring R if and only if in addition to all connected components of P being pure, the pairwise difference in their length is not greater than one. A similar notion to nearly "Gorenstein" is "almost Gorenstein". It is defined in [GMP13] and almost Gorenstein Hibi rings are characterized in [Miy18]. In this work we focus on the reduced structure of Z and are able to give a complete characterization.

1.2 Objective

We provide a complete discrete-geometrical description of the non-Gorenstein locus Z in the toric case in Theorem 3.1. Subsequently, in Theorem 4.2, we apply this result to Hibi rings and characterize Z in terms of non-graded subsets of $P \dot\cup \{\infty, -\infty\}$. A similar description to ours is given in [MP20]. We compare results and provide a new proof to a key result from [MP20] in Chapter 4. Finally, as a non-toric application of Theorem 3.1 the non-Gorenstein loci of the first secant variety of Segre varieties are studied. This is made possible by [MOZ15] where a covering of the secant variety by toric open patches is constructed using methods from statistics. The authors investigate different regularity conditions of the secant varieties, amongst others the Gorenstein and $\mathbb{Q}$-Gorenstein property and smoothness. We extend these results by computing the dimensions of the non-Gorenstein loci.

1.3 Outline

Overall the paper has the following structure:

- Chapter 2: Preliminaries are covered, in particular (toric) divisors, canonical modules and toric varieties are introduced.
- Chapter 3: We describe the non-Gorenstein loci of toric varieties in Theorem 3.1.
- Chapter 4: Hibi rings are defined and the underlying polyhedral cone is constructed. Basic properties are shown and Theorem 4.2 is proven, characterizing the non- Gorenstein locus. We compare results with [MP20] and give a new proof to a key theorem.

- Chapter 5: We introduce secants of Segre varieties, the notation and the objects from [MOZ15]. Finally, by applying results from Chapter 3 the dimension of the non-Gorenstein locus is computed in Corollary 4.1.

2 Preliminaries

In this section we introduce all notions from (toric) algebraic geometry and from homological algebra that are needed.

2.1 Algebraic geometry

Since many sources use the language of schemes we do so as well, but most of the thesis is comprehensible to readers that know classical algebraic geometry. We start by introducing the standard notion of a Weil divisor following [Har77].

Definition 2.1. A variety X is called normal if for every element x of X the stalk $\mathcal{O}_{X,x}$ is integrally closed within its field of fractions.

Let from now on X be a normal variety and $\mathcal{F}$ a quasicoherent $\mathcal{O}_X$-module.

Definition 2.2. A prime divisor of X is a codimension one subvariety. Let $D_1, \dots, D_n$ be prime divisors of X and $a_1, \dots, a_n$ be integers. We call the formal sum

$$D = a_1 D_1 + \cdots + a_n D_n$$

a (Weil) divisor on X. If all integers a_i are nonnegative we call D effective and write $D \geq 0$.

For normal varieties there is a notion of multiplicity of vanishing for rational functions along codimension one subvarieties.

Definition 2.3. Let D be a prime divisor of X with generic point x. Since X is normal, the one-dimensional stalk $\mathcal{O}_{X,x}$ is a regular ring and a discrete valuation ring (see [Har77] Theorem I 6.2.A). We denote the associated valuation

$$\nu_D : K(X) \longrightarrow \mathbb{Z}$$

where $K(X)$ is the function field of X. To an element $f \in K(X)$ we associate the divisor

$$\operatorname{div}(f) = \sum_D \nu_D(f) D$$

where D runs over all prime divisors of X. This is a finite sum since X is normal (see [Har77] Lemma II 6.1).

Definition 2.4. Let

$$D = a_1 D_1 + \cdots + a_n D_n$$

be a divisor on X and U an open subset of X. We define the restriction $D|_U$ of D to U to be the sum over all divisors $a_i(D_i \cap U)$ of U where D_i intersects U nontrivially.

Definition 2.5. To a divisor

$$D = a_1 D_1 + \cdots + a_n D_n$$

we associate the usual $\mathcal{O}_X$-module

$$\mathcal{O}_X(D)(U) := \{f \in K(X) | \ \operatorname{div}(f)|_U + D|_U \geq 0\}.$$

It consists of all rational functions that vanish at least $-a_i$-times along each divisor D_i and have no poles away from the union of all D_i.

Definition 2.6. The dual sheaf $\mathcal{F}^\vee$ is defined to be the sheaf $\mathscr{H}om(\mathcal{F}, \mathcal{O}_X)$ of $\mathcal{O}_X$-module morphisms

$$U \longmapsto \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}|_U, \mathcal{O}_X|_U).$$

We will study the module of global sections of certain $\mathcal{O}_X$ modules. For affine varieties $\mathcal{O}_X$ modules are uniquely determined by their global sections. More precisely, the global sections functor that takes an $\mathcal{O}_X$-module $\mathcal{F}$ to its module of global sections $\Gamma(\mathcal{F})$ induces an equivalence of categories (see Corollary II 5.5, [Har77]). The following proposition is a consequence.

Proposition 2.1. *Let X be affine: $X = Spec(A)$. Then the module $\Gamma(\mathcal{F}^\vee)$ of global sections is the dual module*

$$\Gamma(\mathcal{F})^\vee := \ Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) = Hom_A(\Gamma(\mathcal{F}), A).$$

Proposition 2.2 (See [CLS11] proof of Theorem 6.0.20.)**.** *Let D be a divisor on X. Then*

$$\mathcal{O}_X(D)^\vee = \mathcal{O}_X(-D).$$

2.2 Homological algebra

In this thesis most of the results concern the affine setting, so we may focus on modules instead of sheafs in this section, following [HB98]. We recall that regularity conditions such as the Cohen-Macaulay property and the Gorenstein property are characterized by the existence of so called canonical modules satisfying special properties.

In this section let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of Krull dimension n and M a finite R-module.

Definition 2.7. The injective dimension of M is the minimal length of all bounded injective resolutions $M \to I^\bullet$. If no such resolution exists then the injective dimension is defined to be infinite.

Proposition 2.3 ([HB98], Proposition 3.1.14)**.** *The injective dimension of M is the supremum*

$$\sup\{i \mid Ext^i_R(k, M) \neq 0\}.$$

Definition 2.8. R is Gorenstein if R has finite injective dimension as an R-module.

Definition 2.9. We call M a canonical module of R if

$$\dim_k \mathrm{Ext}^i_R(k, M) = \begin{cases} 1, & \text{if } i = n \\ 0, & \text{if } i \neq n \end{cases}.$$

Proposition 2.4 ([HB98], Theorem 3.2.10.)**.** *If R is Gorenstein then R is a canonical R-module.*

Definition 2.10. R is a Cohen Macaulay ring if its depth equals its dimension n.

Proposition 2.5 ([HB98], Theorem 3.3.4.)**.** *If R is Cohen Macaulay and if M and M' are canonical R-modules then M and M' are isomorphic.*

Proposition 2.6 ([HB98], Proposition 3.1.20.)**.** *We have the following implications:*

$$R \text{ is regular} \implies R \text{ is Gorenstein} \implies R \text{ is Cohen Macaulay}$$

Let from now on R be any Noetherian ring, not necessarily local.

Definition 2.11. R is Gorenstein if each localization $R_\mathfrak{p}$ for $\mathfrak{p}$ a prime ideal is Gorenstein.

Definition 2.12. Let M be an R-module. Then M is a canonical module of R if for each prime $\mathfrak{p}$ the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is canonical.

Theorem 2.1 ([HB98], Theorem 3.3.6.)**.** *Let R be Cohen-Macaulay and the homomorphic image of a Gorenstein ring. Then there is a canonical R-module.*

Remark 2.1. Note that every finitely generated k-algebra is the homomorphic image of a free and hence Gorenstein k-algebra. In particular, every Cohen-Macaulay coordinate ring of an affine variety posseses a canonical module.

2.3 Toric geometry

In this section we introduce basic notions from toric geometry following [CLS11]. The most important objects, toric varieties, can be studied in terms of discrete geometry. The classification of affine, normal toric varieties is reiterated and the canonical modules of these Cohen Macaulay varieties will be defined. We start by fixing some notation:

- N is a free abelian group of rank n with dual $M = \mathrm{Hom}(N, \mathbb{Z})$.
- $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$, $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$ are $\mathbb{R}$-linear spaces of dimension n.
- $\sigma \subseteq N_\mathbb{R}$ is a rational, pointed cone.
- $\sigma^\vee = \{l \in M_\mathbb{R} | \ l(\sigma) \subseteq \mathbb{R}_{\geq 0}\}$.
- $\Sigma(\sigma)$ is the fan in $N_\mathbb{R}$ consisting of the faces of σ.

- $\Sigma(\sigma)[1]$ is the set of rays ρ of σ.
- For every ray $\rho \in \Sigma(\sigma)[1]$, u_ρ denotes the primitive ray-generator.
- $F \subseteq \sigma^\vee$ is a face.
- $F^\perp \subseteq N_\mathbb{R}$ is the orthogonal complement of F.

Definition 2.13. Let G be an abelian semigroup with neutral element. We denote by

$$k[G] := \bigoplus_{g \in G} k \cdot \chi^g$$

the $k-$algebra spanned as a linear space by all monomials χ^g, $g \in G$ and with multiplication uniquely determined by

$$\chi^g \chi^h = \chi^{g+h} \text{ for } g, h \in G.$$

Remark 2.2. Observe that this construction defines a left-adjoint to the forgetful functor from k-algebras to semigroups. In other words, for G a semigroup and A a $k-$algebra the set $\mathrm{Hom}_{\text{k-alg}}(k[G], A)$ of $k-$algebra morphisms and the set $\mathrm{Hom}_{\text{s-groups}}(G, (A, \cdot))$ of semigroup morphisms are in natural bijection. Here A is considered a semigroup via multiplication. This bijection is induced by the map

$$\begin{aligned} \mathrm{Hom}_{\text{s-groups}}(G, (A, \cdot)) &\longrightarrow \mathrm{Hom}_{\text{k-alg}}(k[G], A) \\ \gamma &\longmapsto (\chi^g \mapsto \gamma(g)) \end{aligned}$$

since a morphism $k[G] \to A$ of k-algebras is uniquely determined by the images of the monomials. In particular, the k-valued points $\mathrm{Hom}_{\text{k-alg}}(k[G], k)$ of $\mathrm{Spec} k[G]$ are in natural bijection with semigroup-morphisms $G \to k$, where k is a semigroup via multiplication.

Definition 2.14. The n-dimensional algebraic torus T^n is the variety $\mathrm{Spec}(k[M])$ where M is the free abelian group of rank n. T^n is an algebraic group via the natural map

$$\begin{aligned} k[M] &\longrightarrow k[M] \otimes k[M] \\ \chi^m &\longmapsto \chi^m \otimes \chi^m. \end{aligned}$$

Remark 2.3. By choosing a basis of M one identifies $k[M]$ with the ring of Laurent-polynomials $k[\mathbb{Z}^n] = k[x_1^\pm, \dots, x_n^\pm]$. The k-valued points of $k[x_1^\pm, \dots, x_n^\pm]$ are in bijection with elements of $(k^*)^n$.

The adjunction from remark 2.2 can be made concrete in this example: a group-morphism $\gamma : \mathbb{Z}^n \to (k, \cdot)$, defines a morphism of k-algebras

$$\begin{aligned} k[x_1^\pm, \dots, x_n^\pm] &\longrightarrow k \\ x_1^{d_1} \cdots x_n^{d_n} &\longmapsto \gamma(d_1, \dots, d_n). \end{aligned}$$

The images of the unit vectors $\gamma(e_i)$ are arbitrary elements of k^* and determine both γ and the k-valued point uniquely.

Definition 2.15. A toric variety of dimension n is a variety X together with a group-action of the algebraic torus

$$T^n \times X \longrightarrow X$$

such that there is an open, dense orbit.

Proposition 2.7. *$X := \mathit{Spec}(k[\sigma^\vee \cap M])$ is a toric variety.*

Proof. Since σ is a rational polyhedral cone, the semigroup $\sigma^\vee \cap M$ is finitely generated (see Gordan's Lemma [CLS11] Proposition 1.2.17). Consequently $k[\sigma^\vee \cap M]$ can be seen to be an integral, finitely generated k-algebra defining a variety. The map of k-algebras

$$\begin{aligned} k[\sigma^\vee \cap M] &\longrightarrow k[\sigma^\vee \cap M] \otimes k[M] \\ \chi^m &\longmapsto \chi^m \otimes \chi^m \end{aligned}$$

defines a group-action

$$\mathrm{Spec}(k[M]) \times \mathrm{Spec}(\sigma^\vee \cap M) \longrightarrow \mathrm{Spec}(\sigma^\vee \cap M)$$

and the orbit of the $k-$valued point

$$\begin{aligned} x : \sigma^\vee \cap M &\longrightarrow k \\ m &\longmapsto 1 \end{aligned}$$

is the image of the map $\mathrm{Spec}(k) \times T^n \to X \times T^n \to X$. The associated map of k-algebras

$$\begin{aligned} k[\sigma^\vee \cap M] &\longrightarrow k \otimes_k k[M] \\ \chi^m &\longmapsto x(m) \otimes \chi^m \end{aligned}$$

is the natural inclusion of $k[\sigma^\vee \cap M]$ in $k[M]$. Since $k[M]$ is obtained from $k[\sigma^\vee \cap M]$ by localizing along any monomial χ^m where m is in the interior of $\sigma^\vee$, the orbit $T^n.x$ is a principal open subset of X. $\square$

It is well known that all normal, affine toric varieties are of the form as described in Proposition 2.7.

Theorem 2.2 (see [CLS11] Theorem 1.3.5)**.** *Let X be an affine toric variety. Then X is normal if and only if X is of the form $\mathit{Spec}(k[\sigma^\vee \cap M])$ where σ is a pointed rational polyhedral cone.*

Crucially, the combinatorics of σ reflect in the geometry of X: the faces of $\sigma^\vee$ are in inclusion preserving bijection with the torus-invariant subvarieties of X. This result is called orbit-cone correspondence and since torus-invariant subvarieties are defined by graded primes we use the following phrasing.

Theorem 2.3 (Orbit-cone correspondence, see [CLS11] Theorem 3.2.6)**.** *The map*

$$\begin{aligned} \{\text{faces of } \sigma^\vee\} &\longrightarrow \{M\text{-graded primes of } k[\sigma^\vee \cap M]\} \\ F &\longmapsto \langle \chi^m |\ m \in (\sigma^\vee \cap M) \setminus F \rangle \end{aligned}$$

is an inclusion-reversing bijection. The dimension of F is the dimension of the associated ideal.

Since we consider normal varieties there is the usual theory of Cartier and Weil divisors (consider [Har77] Chapter two). We introduce some basic notation concerning torus-invariant divisors.

By the orbit-cone correspondence the following definition provides a complete list of all torus-invariant hypersurfaces:

Definition 2.16. Let ρ be a ray of $\sigma^\vee$ and u_ρ the primitive ray generator. D_ρ denotes the prime-divisor defined by the ideal I_ρ which is spanned by all monomials away from the facet defined by ρ:

$$I_\rho := \langle \chi^m |\ m \in \sigma^\vee \cap M,\ \langle u_\rho, m \rangle > 0 \rangle.$$

Definition 2.17. We call a divisor which is a sum of torus-invariant prime divisors

$$D = \sum_{\rho \in \Sigma(\sigma)(1)} a_\rho D_\rho$$

a torus-invariant divisor and associate to it the polyhedron

$$P_D := \{m \in M_\mathbb{R} |\ \forall \rho \in \Sigma(\sigma)(1) : \langle u_\rho, m \rangle \geq -a_\rho\}$$

with recession cone $\sigma^\vee$.

Proposition 2.8 ([CLS11] Proposition 4.3.3.)**.** *Let D be a torus-invariant divisor. The global sections $\Gamma(\mathcal{O}_X(D))$ are spanned as a k-linear space by all monomials lying in P_D.*

$$\Gamma(\mathcal{O}_X(D)) = \bigoplus_{m \in P_D \cap M} k \cdot \chi^m.$$

Definition 2.18. Let $R = k[\sigma^\vee \cap M]$ be a normal toric ring. The torus-invariant divisor

$$K = \sum_{\rho \in \Sigma(\sigma)(1)} -D_\rho$$

is the negative sum of all torus-invariant hypersurfaces D_ρ. It is called the canonical divisor.

In the following study of normal affine toric rings we take advantage of the fact that they are Cohen Macaulay ([CLS11] Theorem 9.2.9.) and hence possess a canonical module (Theorem 2.1).

Theorem 2.4. *The module*
$$\omega = \Gamma(\mathcal{O}_X(K))$$
of global sections of $\mathcal{O}_X(K)$ *is a canonical* R*-module.*

We only give a sketch of a proof. $\mathcal{O}_X(K)$ is shown in [CLS11] to be the extension of the canonical sheaf ω_0 of the smooth locus U_0 to X. This extensions defines a canonical module for Cohen Macauly varieties such as normal toric varieties. For more on this consider [Shi14], Chapter 5.

3 Toric rings

In this chapter we compute the non-Gorenstein locus Z of normal toric varieties. We prove that Z is the vanishing locus of some homogeneous radical ideal $\sqrt{\operatorname{tr}(\omega)}$ and compute a decomposiution of Z into irreducible components by determining the minimal primes lying over $\operatorname{tr}(\omega)$. These are homogeneous primes and as such correspond to certain faces of $\sigma^\vee$.

Definition 3.1. We define the trace ideal of a module M of R to be

$$\operatorname{tr}(M) := \sum_{\varphi \in \operatorname{Hom}_R(M,R)} \varphi(M).$$

Note that $\operatorname{tr}(M)$ is an ideal. Indeed, it is the submodule of R that forms the direct limit of all R-module maps $M \longrightarrow R$. The radical of the trace ideal of ω describes the non-Gorenstein locus:

Lemma 3.1 ([HHS19], Lemma 2.1)**.** *For every prime* $\mathfrak{p} \in Spec(R)$*,* $R_\mathfrak{p}$ *is not Gorenstein if and only if* $tr(\omega) \subseteq \mathfrak{p}$*.*

Proof. For the first implication assume that $\operatorname{tr}(\omega)$ is not contained in $\mathfrak{p}$. Then the localization $\operatorname{tr}(\omega)_\mathfrak{p}$ is $R_\mathfrak{p}$. Since both operations, taking the trace and localizing at $\mathfrak{p}$ are direct limits, they commute:
$$\operatorname{tr}(\omega_\mathfrak{p}) = \operatorname{tr}(\omega)_\mathfrak{p} = R_\mathfrak{p}.$$
For this equality to hold there needs to be a map $\varphi : \omega_\mathfrak{p} \longrightarrow R_\mathfrak{p}$ whose image is not contained in the maximal ideal $\mathfrak{p}R_\mathfrak{p}$. φ is surjective and hence $\omega_\mathfrak{p}$ splits:

$$\omega_\mathfrak{p} = R_\mathfrak{p} \oplus W$$

holds for some $R_\mathfrak{p}$-module W. $\operatorname{Ext}^i_{R_\mathfrak{p}}(k, \cdot)$ is an additive functor and hence

$$\operatorname{Ext}^i_{R_\mathfrak{p}}(k, R_\mathfrak{p}) \oplus \operatorname{Ext}^i_{R_\mathfrak{p}}(k, W) = \operatorname{Ext}^i_{R_\mathfrak{p}}(k, R_\mathfrak{p} \oplus W) = \operatorname{Ext}^i_{R_\mathfrak{p}}(k, \omega_\mathfrak{p}).$$

By definition of the dualizing module $\operatorname{Ext}^i_{R_\mathfrak{p}}(k, \omega_\mathfrak{p})$ vanishes for all i except for at most one. Consequently $\operatorname{Ext}^i_{R_\mathfrak{p}}(k, R_\mathfrak{p})$ vanishes for all i except for one which by Proposition

2.3 implies that $R_\mathfrak{p}$ has finite injective dimension and hence is a canonical $R_\mathfrak{p}$-module.

For the other direction let $R_\mathfrak{p}$ be Gorenstein. By Proposition 2.4 $R_\mathfrak{p}$ is a canonical $R_\mathfrak{p}$-module and hence $\omega_\mathfrak{p}$ is isomorpic to $R_\mathfrak{p}$ by Proposition 2.5. The ideal $\mathrm{tr}(\omega_\mathfrak{p})$ equals $R_\mathfrak{p}$. For the $R_\mathfrak{p}$-ideal $\mathrm{tr}(\omega)R_\mathfrak{p}$ it holds

$$\mathrm{tr}(\omega)R_\mathfrak{p} = \mathrm{tr}(\omega)_\mathfrak{p} = \mathrm{tr}(\omega_\mathfrak{p}) = R_\mathfrak{p}$$

and hence $\mathrm{tr}(\omega)$ is not contained in $\mathfrak{p}$. □

Remark 3.1. This proposition implies that the non-Gorenstein locus is the vanishing set of the radical $\sqrt{\mathrm{tr}(\omega)}$. Note that the radical $\sqrt{\mathrm{tr}(\omega)}$ is the intersection of all primes containing $\mathrm{tr}(\omega)$.

The trace ideal $\mathrm{tr}(\omega)$ is a homogeneous ideal with generators described in the following proposition. Recall that ω is spanned by all monomials in the polyhedron

$$P_K = \{x \in M_\mathbb{R} |\ \forall \rho \in \Sigma(\sigma)(1) : \langle u_\rho, x\rangle \geq 1\}$$

which is obtained by translating all facets of $\sigma^\vee$ into the interior by one lattice-length.

Proposition 3.1. *$tr(\omega) = \langle \chi^{m+m'} |\ m \in P_K \cap M, m' \in P_{-K} \cap M\rangle$. In particular, $tr(\omega)$ is $M-$graded.*

Proof. By Theorem 2.4 ω is the module of global sections of the sheaf $\mathcal{O}(K)$ associated to the canonical divisor $\sum_{\rho\in\Sigma(\sigma)(1)} -D_\rho$. Hence its dual $\omega^\vee$ is the module of global sections $\Gamma(\mathcal{O}(-K))$ by propositions 2.1 and 2.2:

$$\omega^\vee = \mathrm{Hom}_R(\omega, R) = \Gamma(\mathcal{O}(-K)) = \bigoplus_{\mathrm{div}(\chi^m)-K\geq 0} k\cdot\chi^m = \bigoplus_{m\in P_{-K}\cap M} k\cdot\chi^m.$$

Both ω and $\omega^\vee$ are submodules of the function field $K(X)$. Evaluation of an element of $\sigma^\vee$ at an element of σ is multiplication within $K(X)$ and the statement of the proposition follows. □

We aim to compute the irreducible components of the non-Gorenstein locus Z. By Lemma 3.1, Z is the vanishing locus of the radical $\sqrt{\mathrm{tr}(\omega)}$, and consequently the irreducible components are the vanishing loci of the minimal primes lying over $\mathrm{tr}(\omega)$. Since $\mathrm{tr}(\omega)$ is M-graded, every minimal prime $\mathfrak{p}$ is as well:

Lemma 3.2. *Let $A = \bigoplus_{\underline{d}\in\mathbb{Z}^d} A_{\underline{d}}$ be a $\mathbb{Z}^d$-graded ring and $I \subseteq A$ a graded ideal. Then every prime $\mathfrak{p}$ lying over I that is minimal with this property is graded.*

Proof. It suffices to prove that $A/\mathfrak{p}$ is graded. By the equality

$$A/\mathfrak{p} = (A/I)/(\mathfrak{p}/I)$$

it suffices to show that $\mathfrak{p}$ mod I is a graded ideal in A/I. By redefining A and $\mathfrak{p}$ we may assume $I = 0$. Let J be the graded ideal generated by all homogeneous elements of $\mathfrak{p}$. By minimality of $\mathfrak{p}$, checking that J is prime shows equality and finishes the proof.

Consider any product fg of elements in A that lies in J. It is to show that f or g lie in J as well. Since $\mathfrak{p}$ is prime we may assume without loss of generality that it contains f. We proceed by induction on the sum of the number of monomials terms appearing in f and g respectively. f and g vanish if this number is zero so the induction start is clear. We fix a monomial ordering and pick minimal monomials χ^w appearing in f and $\chi^{w'}$ appearing in g. By construction, $\chi^{w+w'}$ appears in fg. Since J is a homogeneous ideal it contains $\chi^{w+w'}$, and hence χ^w or $\chi^{w'}$ lie in $\mathfrak{p}$. By definition of J it contains this monomial. Consequently we either obtain that the product $(f - \chi^w)g$ lies in J or that the product $f(g - \chi^{w'})$ is contained in J. In each case, appling the induction hypothesis yields that one of the factors is in J which finishes the proof. □

We now classify the graded primes that contribute to the non-Gorenstein locus. Let $\mathfrak{p}$ be a graded prime. By the orbit cone correnspondence (Theorem 2.3) $\mathfrak{p}$ is defined by a face F of $\sigma^\vee$:

$$\mathfrak{p} = \langle \chi^m | \; m \in \sigma^\vee \setminus F \cap M \rangle.$$

Definition 3.2. We define

$$F[1] := \{x \in M_{\mathbb{R}} | \; \forall \rho \in \Sigma(\sigma)(1), \; u_\rho \in F^\perp : \langle u_\rho, x \rangle = 1\}$$

to be the affine-linear space that is the intersection of all facet-defining hyperplanes containing F, translated by one lattice-length into the interior of $\sigma^\vee$.

The following lemma characterizes the graded primes lying over $\mathrm{tr}(\omega)$ in terms of F:

Lemma 3.3. *Let $\mathfrak{p} \subseteq R$ be a graded prime defined by a face F of $\sigma^\vee$. Then $tr(\omega) \subseteq \mathfrak{p}$ holds if and only if $F[1]$ does not contain a lattice point.*

Proof. For the first implication let z be a lattice point in $F[1]$. Let w be a lattice point in the relative interior of F. In other words, for every primitive generator u_ρ of σ the following inequalities hold.

$$\langle u_\rho, w \rangle > 0, \text{ if } u_\rho \notin F^\perp$$
$$\langle u_\rho, w \rangle = 0, \text{ if } u_\rho \in F^\perp.$$

After replacing w with a positive integer multiple we may assume

$$\langle u_\rho, w \rangle > \langle u_\rho, z \rangle + 1$$

to hold for all u_ρ not in $F^\perp$. Then $w - z \in P_{-K}$ and $z + w \in P_K$. Hence

$$2w = (w - z) + (z + w) \in P_{-K} \cap M + P_K \cap M.$$

By Proposition 3.1 χ^{2w} lies in $\sqrt{\mathrm{tr}(\omega)}$. χ^{2w} is not contained in $\mathfrak{p}$, and hence the radical of $\mathrm{tr}(\omega)$ is not contained in $\mathfrak{p}$. Thus $\mathrm{tr}(\omega)$ is not contained either.

For the other direction let $\chi^w \in \mathrm{tr}(\omega) \setminus \mathfrak{p}$. Then w lies in F, and by Proposition 3.1 there is an element $z \in P_K \cap M$ with $w - z \in P_{-K}$:

$$-\langle u_\rho, z\rangle = \langle u_\rho, w - z\rangle \geq -1$$

holds for all u_ρ in $F^\perp$. So $\langle u_\rho, z\rangle \leq 1$ and hence $\langle u_\rho, z\rangle = 1$ since $z \in P_K$. We obtain $z \in F[1]$. □

Remark 3.2. The above lemma states that the vanishing locus of a graded prime $\mathfrak{p}$ defined by a face F is contained in the non-Gorenstein locus if and only if $F[1]$ does not contain a lattice point.

Although it won't be of relevance in this thesis, lemma 3.3 can easily be generalized to normal toric varieties that are not affine. We use the notation from [CLS11]. Let X be a normal toric variety defined by a fan Σ of rational, pointed cones in $N_\mathbb{R}$, $Y \subseteq X$ a nonempty torus-invariant subariety given by a cone $\sigma \in \Sigma$.

Theorem 3.1. *The torus-invariant subvariety Y is contained in the non-Gorenstein locus Z if and only if there is no element m of M such that*

$$\langle u_\rho, m\rangle = 1$$

holds for every ray ρ of σ.

Proof. Both Z and Y are closed. Since Y intersects the affine open $\mathrm{Spec}(k[\sigma^\vee \cap M])$ in the set $\mathrm{Spec}(k[\sigma^\perp \cap M])$ we may replace Y with $\mathrm{Spec}(k[\sigma^\perp \cap M])$, X with $\mathrm{Spec}(k[\sigma^\vee \cap M])$, and Z with the non-Gorenstein locus of X.

By Lemma 3.1 and Lemma 3.3, Y is contained in Z if and only if for the choice $F = \sigma^\perp$, $F[1]$ does not contain a lattice point. In other words, if there is no element m of M with

$$u_\rho \in (\sigma^\perp)^\perp : \langle u_\rho, m\rangle = 1$$

for every ray ρ of σ. All ray generators of σ are contained in

$$(\sigma^\perp)^\perp = \sigma - \sigma.$$

□

4 Hibi rings

In this chapter we introduce Hibi rings and compute their non-Gorenstein loci. We start by introducing some basic notions of the underlying combinatorial objects: partially ordered sets.

4.1 Partially ordered sets

Definition 4.1. We call a set P together with a transitive, reflexive order $\leq$ a partially ordered set, or poset. We call P connected if the symmetric hull of the relation $\leq$ has only one equivalence class.

Definition 4.2. Let $a \leq b$ be two elements of P. We call the set

$$[a, b] := \{x \in P |\ a \leq x \leq b\}$$

consisting of all elements that lie between a and b an interval and call P bounded if there are a minimal element a and a maximal element b with

$$P = [a, b].$$

Definition 4.3. For different elements $a \leq b$ of a partially ordered set the covering relation denoted by $a \lessdot b$ is defined to hold if there are no elements lying properly between a and b:

$$\#[a, b] = 2.$$

Here # denotes the cardinality of a set.

Definition 4.4. We call a totally ordered poset

$$a_1 < \cdots < a_r$$

of cardinality r a chain of length $r - 1$. P is called pure if all chains contained in P that are maximal with respect to inclusion have the same length.

Definition 4.5. The set $\mathcal{I}(P)$ of subsets $I \subseteq P$ that are closed from below:

$$\forall a \in P,\ i \in I : a \leq i \implies a \in I$$

is called the lattice of order ideals.

Remark 4.1. Lattices are partially ordered sets in which unique suprema and infima exist. They are both combinatorial and algebraic objects. Supremum and infimum in $\mathcal{I}(P)$ are just intersection and union respectively and satisfy the following rules:

$$I \cup (J \cap H) = (I \cup J) \cap (I \cup H)$$
$$I \cap (J \cup H) = (I \cap J) \cup (I \cap H).$$

Lattices that satisfy these rules are called distributive lattices. In fact, every finite distributive lattice is the lattice of order ideals of some partially ordered set. This result is called Birkhoff's representation theorem [Bir37] and asserts that finite distributive lattices and finite partially ordered sets are essentially the same objects.

Definition 4.6. Let P be a poset. By $\overline{P}$ we denote the bounded poset $P \dot\cup \{\infty, -\infty\}$ where ∞ and $-\infty$ are the new maximal the new minimal element respectively.

4.2 Basic properties of Hibi rings

We finally introduce the objects of interest:

Definition 4.7. Let P be a finite poset and let $k[t, x_p,\ p \in P]$ be the free k-algebra in the variables x_p for all p in P and the variable t. For each poset ideal I we denote

$$x^I := \prod_{p \in I} x_p.$$

The k-algebra

$$R(P) := k[tx^I, I \in \mathcal{I}(P)] \subset k[t, x_p,\ p \in P]$$

generated by all monomials tx^I where I runs over all poset-ideals is the Hibi-ring associated to P.

Example 4.1. Let P be the totaly ordered set with n elements $p_1 \leq \cdots \leq p_n$. Then the associated Hibi ring

$$R(P) = k[t, tx_{p_1}, tx_{p_1}x_{p_2}, \ldots, tx_{p_1} \cdots x_{p_n}]$$

has algebraically independent generators and is isomorphic to the free k-algebra in $n+1$ varaibles.

Definition 4.8. We denote by

$$C(P) := \{\psi : \overline{P} \longrightarrow \mathbb{R} |\ \forall a, b \in \overline{P} : a \leq b \implies \psi(a) \geq \psi(b)\ , \psi(\infty) = 0\}.$$

the cone of order-reversing maps from

$$\overline{P} = P \dot{\cup} \{-\infty, \infty\}$$

to $\mathbb{R}$ taking 0 as minimal value. It is a fulldimensional subcone of the $\mathbb{R}$-linear space of maps

$$M_{\mathbb{R}} = \{\psi : \overline{P} \longrightarrow \mathbb{R} |\ \psi(\infty) = 0\}.$$

By M we denote the sublattice of maps that take values only in the integers. The cone $C(P)$ is the cone over the lattice polytope

$$Q(P) := \{\psi \in C(P) |\ \psi(-\infty) = 1\}$$

consisting of the surjective order reversing maps from $\overline{P}$ to the real interval $[0, 1]$.

Remark 4.2. Consider a defining inequality

$$\psi(a) \geq \psi(b)$$

of $C(P)$ given by elements $a \leq b$ in $\overline{P}$. It defines a face

$$F_{a \leq b} := \{\psi \in C(P) |\ \psi(a) = \psi(b)\}$$

of $C(P)$. For every strict inclusion of intervals

$$[c,d] \subset [a,b]$$

there is a strict inclusion of faces

$$F_{a \leq b} \subset F_{c \leq d}.$$

Consequently the maximal proper faces of $C(P)$ are the faces

$$F_{a \lessdot b} := \{\psi \in C(P) |\ \psi(a) = \psi(b)\}$$

defined by covering relations. This defines a bijection between the family of facets and the covering relations in $\overline{P}$.

Theorem 4.1. *$R(P)$ is the normal toric ring*

$$k[C(P) \cap M]$$

associated to $C(P)$ and in particular of Krull-dimension $\#P + 1$.

Proof. By definition, $R(P)$ is the $k-$algebra $k[G]$ associated to the multiplicative monoid G generated by all monomials tx^I. These monomials are in natural bijection to the lattice points of $Q(P)$: consider the embedding Θ of monoids induced by

$$\begin{aligned} \Theta : G &\longrightarrow C(P) \cap M \\ tx^I &\longmapsto \Theta(tx^I) \end{aligned}$$

where

$$\begin{aligned} \Theta(tx^I) : \overline{P} &\longrightarrow \mathbb{Z} \\ p &\longmapsto \begin{cases} 1, & \text{for } p \in I \cup \{-\infty\} \\ 0, & \text{for } p \notin I \cup \{-\infty\} \end{cases} \end{aligned}$$

is a lattice point of the polytope $Q(P)$. Conversely, let

$$\psi : \overline{P} \longrightarrow \{0,1\}$$

be a surjective, order-reversing map: ψ is a lattice point of $Q(P)$. Then $\psi^{-1}(1) \cap P$ is an order-ideal and ψ is the image of tx^I. In fact, G is the submonoid of $C(P) \cap M$ generated by all lattice points of $Q(P)$.

We now prove equality of $C(P) \cap M$ and G, finishing the proof. For this let

$$\psi : \overline{P} \longrightarrow \mathbb{Z}$$

be an order-reversing map taking minimal value 0 and maximal value $m > 0$. By induction on m we show that ψ lies in G. If m is one then ψ is a lattice point of $Q(P)$, showing the induction start. Let now m be arbitrary. Observe that $\psi^{-1}(m) \cap P$ is an order-ideal and denote it I. $\psi - \Theta(tx^I)$ is an element of $C(P) \cap M$ and by induction hypothesis an element of G. $\square$

Definition 4.9. Let $\phi : \overline{P} \twoheadrightarrow P'$ be a surjective, order preserving map of partially ordered sets with connected fibres. We denote by F_ϕ the face

$$F_\phi := \bigcap_{\substack{a \lessdot b, \\ \phi(a)=\phi(b)}} F_{a \lessdot b}$$

consisting of the functions $\psi \in C(P)$ that are constant on the fibres of ϕ.

The faces of $C(P)$ can be characterized in terms of equivalence classes of such maps. Two maps $\phi : \overline{P} \twoheadrightarrow P'$ and $\tilde{\phi} : \overline{P} \twoheadrightarrow \tilde{P}$ are identified if there is a set-theoretic bijection $P' \longrightarrow \tilde{P}$ that makes the diagram

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "\tilde{\phi}"] & P' \\ \tilde{P} \arrow[ur] & \end{tikzcd}$$

commute.

Proposition 4.1. *The map*

$$\Theta : \phi \longmapsto F_\phi = \bigcap_{\substack{a \lessdot b, \\ \phi(a)=\phi(b)}} F_{a \lessdot b}$$

induces a bijection between the set of equivalence classes of surjective, order-preserving maps of posets $\phi : \overline{P} \twoheadrightarrow P'$ with connected fibres, and the set of faces of $C(P)$.

Proof. Let F be a face of $C(P)$. To define an inverse to Θ let $\sim$ on $\overline{P}$ be the equivalence relation on $\overline{P}$ generated by all identifications $a \sim b$ where $a \lessdot b$ and $F \subseteq F_{a \lessdot b}$. Note that every element ψ of F is constant on equivalence classes. $\overline{P}/\sim$ is a partially ordered set where for equivalence classes $\overline{a}, \overline{b}$ we define $\overline{a} \leq \overline{b}$ on $\overline{P}$ to hold if for every element ψ of F it holds $\psi(\overline{a}) \leq \psi(\overline{b})$.

The natural map from $\overline{P}$ to $\overline{P}/\sim$ is order preserving and by construction of $\sim$ it has connected fibres. We denote it ϕ and aim to show $\Theta(\phi) = F$. Observe that exactly those covering relations $a \lessdot b$ are contained in a fibre of ϕ for which we have the inclusion $F \subseteq F_{a \lessdot b}$. Hence by definition of Θ, $\Theta(\phi)$ is contained in exactly those facets $F_{a \lessdot b}$ that contain F, proving equality.

For the converse statement we start with a surjective, order-preserving map

$$\phi : \overline{P} \twoheadrightarrow P'$$

with connected fibres and denote by $\sim$ the equivalence relation assoctiated to the face $\Theta(\phi)$. It is to show that P' and $P/\sim$ are in bijection relative P. Towards constructing the desired map we prove that every fibre $\phi^{-1}(p')$ is an equivalence class of $\sim$. Since the fibre is connected, by construction of $\sim$ all elements lie in the same equivalence class. On the other hand ϕ is constant on equivalence classes, so overall P' and $\overline{P}/\sim$ are in bijection. $\square$

Remark 4.3. Applying the orbit-cone correspondence and the above proposition gives a bijection between the graded primes of $k[C(P) \cap M]$ and surjective, order-preserving maps $\phi : \overline{P} \longrightarrow P'$ with connceted fibres. The prime $\mathfrak{p}$ associated to F_ϕ is generated by those monomials whose exponent is nonconstant on some fibre of ϕ:

$$\mathfrak{p} = \langle \chi^\psi |\ \psi \in C(P) \cap M,\ \psi \text{ is not constant on a fibre of } \phi \rangle.$$

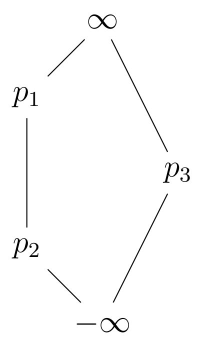
Precomposition with ϕ defines a bijection between those order preserving maps from P' to $\mathbb{R}$ that take minimal value 0, and the face F_ϕ. Consequently F_ϕ is a cone of dimension $\#P' - 1$ and this is the dimension of the zero locus of $\mathfrak{p}$ as well.

Example 4.2. Consider the partially ordered set

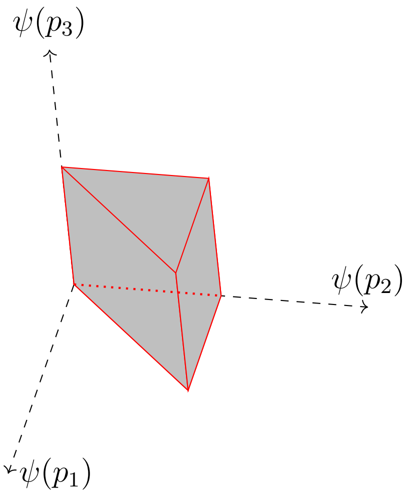
$$P = \{p_1, p_2, p_3\}$$

with only the relation $p_1 \geq p_2$:

$\overline{P}$:



The polytope $Q(P)$ is the three-dimensional polytope consisting of all order-reversing maps from P to the real interval $[0, 1]$:



There are nine one-dimensional faces of $Q(P)$ corresponding to order-preserving surjective maps

$$\phi : \overline{P} \longrightarrow P' = \begin{array}{c} q_2 \\ | \\ q_1 \\ | \\ q_0 \end{array}$$

where P' is the unique bounded partially ordered set with three elements. We give a list that matches the one-dimensional faces F to the fibres of ϕ.

Nr.	F	$\phi^{-1}(q_2)$	$\phi^{-1}(q_1)$	$\phi^{-1}(q_0)$
1	$\mathrm{conv}(\{(0,0,0),\ (0,0,1)\})$	$\{\infty, p_1, p_2\}$	$\{p_3\}$	$\{-\infty\}$
2	$\mathrm{conv}(\{(0,0,0),\ (0,1,0)\})$	$\{\infty, p_1, p_3\}$	$\{p_2\}$	$\{-\infty\}$
3	$\mathrm{conv}(\{(0,0,0),\ (1,1,0)\})$	$\{\infty, p_3\}$	$\{p_1, p_2\}$	$\{-\infty\}$
4	$\mathrm{conv}(\{(0,0,1),\ (0,1,1)\})$	$\{p_1, \infty\}$	$\{p_2\}$	$\{p_3, -\infty\}$
5	$\mathrm{conv}(\{(0,0,1),\ (1,1,1)\})$	$\{\infty\}$	$\{p_1, p_2\}$	$\{p_3, -\infty\}$
6	$\mathrm{conv}(\{(0,1,0),\ (0,1,1)\})$	$\{p_1, \infty\}$	$\{p_3\}$	$\{p_2, -\infty\}$
7	$\mathrm{conv}(\{(0,1,0),\ (1,1,0)\})$	$\{p_3, \infty\}$	$\{p_1\}$	$\{p_2, -\infty\}$
8	$\mathrm{conv}(\{(0,1,1),\ (1,1,1)\})$	$\{\infty\}$	$\{p_1\}$	$\{p_3, p_2, -\infty\}$
9	$\mathrm{conv}(\{(1,1,0),\ (1,1,1)\})$	$\{\infty\}$	$\{p_3\}$	$\{p_1, p_2, -\infty\}$

4.3 The non-Gorenstein locus

Applying the results from Chapter 3 now allows us to characterize the non-Gorenstein locus in terms of non-graded subsets of P:

Definition 4.10. Let P be a finite poset. We call P graded if there is an order-reversing map $\psi : P \longrightarrow \mathbb{Z}$ such that for every covering relation $a \lessdot b$ it holds $\psi(a) = \psi(b) + 1$. Then ψ is called a grading of P.

Let the face F_ϕ of $C(P)$ be defined by a map $\phi : \overline{P} \twoheadrightarrow P'$ as in definition 4.9.

Lemma 4.1. $F_\phi[1]$ *contains a lattice point if and only if every fibre of* ϕ *is graded.*

Proof. By definition, F_ϕ is the intersection

$$F_\phi = \bigcap_{\substack{a \lessdot b, \\ \phi(a)=\phi(b)}} F_{a \lessdot b}$$

of all facets $F_{a \lessdot b}$ where a and b are elements in the same fibre. Consequently the lattice points of

$$F_\phi[1] = \{\psi \in M_\mathbb{R} |\ \forall a \lessdot b,\ \phi(a) = \phi(b) :\ \psi(a) = \psi(b) + 1\}$$

are those functions ψ that define a grading of the fibres of ϕ. This observation proves the first implication.

For the other implication, assume that each fibre of ϕ is graded. Denote for each element q of P' by

$$\psi_q : \phi^{-1}(q) \longrightarrow \mathbb{Z}$$

a grading of the respective fibre. The maps ψ_q together form an element of $F_\phi[1] \cap M$. $\square$

Let now $\mathfrak{p}$ be the graded prime defined by F_ϕ, as in remark 4.3.

Theorem 4.2. *The local ring $k[tx^I, I \in \mathcal{I}(P)]_\mathfrak{p}$ is Gorenstein i.e. the vanishing locus of the graded ideal $\mathfrak{p}$ is not contained in the non-Gorenstein locus if and only if all fibres of ϕ are graded.*

Proof. Apply Lemma 3.1, Lemma 3.3 and Lemma 4.1. $\square$

In the remainder of this chapter we characterize the maximal components of Z.

Definition 4.11. We call a connected subset $A \subseteq \overline{P}$ complete if for all inequalities $a \leq b \leq c,\ a, c \in A$ it holds $b \in A$.

Proposition 4.2. *Let $A \subseteq \overline{P}$ be complete. Then there is a poset $\overline{P}/A$ together with a surjective map of posets $\phi : \overline{P} \longrightarrow \overline{P}/A$ such that the only possibly nontrivial fibre is A.*

Proof. To construct $\overline{P}/A$ let the relation $\preceq$ on $\overline{P}$ be the transitive hull of all relations $a \preceq b$ that hold if either $a \leq b$ holds or if

$$\exists x, y \in A \mid a \leq x, y \leq b.$$

We obtain $\overline{P}/A$ by identifying all elements a, b with $a \preceq b$ and $b \preceq a$ and ϕ is the natural map. All elements of A lie in the same fibre. Conversely from $a \preceq b, b \preceq a$ and $a \neq b$ it follows that there are elements $x, y' \in A$ with $a \leq x$ and $y' \leq a$. By completeness of A it follows $a \in A$, showing that all elements lying in a nontrivial fibre need to be in A. $\square$

Remark 4.4. Complete subsets are exactly the connected fibres of poset morphisms. Consequently a monomial χ^ψ is contained in $\sqrt{\operatorname{tr}(\omega)}$ if and only if ψ is nonconstant on every non-graded complete subset A of $\overline{P}$ by Theorem 4.2 and remark 4.3.

Together with Theorem 4.2 we obtain the following result:

Theorem 4.3. *The maximal components of the non-Gorenstein locus Z are defined by the maps*

$$\phi : \overline{P} = P \dot\cup \{-\infty, \infty\} \longrightarrow \overline{P}/A$$

where A is a minimal non-graded complete subsets of $\overline{P}$.

Proof. Let Z be the non-Gorenstein locus and $\phi' : \overline{P} \twoheadrightarrow P'$ define a maximal irreducible component of Z. If Z is nonempty there is a non-graded fibre

$$A := \phi'^{-1}(p')$$

and the natural map

$$\phi : \overline{P} \longrightarrow \overline{P}/A.$$

factors through ϕ'. Consequently the subset of the non-Gorenstein locus defined by ϕ contains the one defined by ϕ' which shows equality.

Conversely, if A is a minimal non-graded complete subset of $\overline{P}$ then the corresponding irreducible subset of Z can be shown to be maximal by the same reasoning: let $\phi' : \overline{P} \twoheadrightarrow P'$ define an irreducible subset of Z containing the one defined by A. Then the natural map $\overline{P} \longrightarrow \overline{P}/A$ factors through ϕ'. There can be only one nontrivial fibre of ϕ' and it is contained in A. By minimality of A we obtain equality. □

Corollary 4.1.

$$\dim(Z) = \max\{\#P - \#A + 2\}$$

where A runs over all non-graded complete subsets of $\overline{P}$.

Proof. As noted in remark 4.3 the dimension of the face F defined by

$$\phi : \overline{P} \longrightarrow \overline{P}/A$$

is $\#\overline{P}/A - 1$. The equality

$$\#\overline{P}/A = \#\overline{P} - \#A + 1 = \#P - \#A + 3$$

finishes the proof. □

Corollary 4.2. *The codimension of Z is at least 4.*

Proof. The Krull dimension of $k[C(P) \cap M]$ is $\#P + 1$. Since every non-graded poset $A \subseteq \overline{P}$ has at least 5 elements, Corollary 4.1 yields the desired inequality. □

Remark 4.5. As a consequence of Corollary 4.1 we obtain the already known characterizations of Gorenstein Hibi rings and Hibi rings that are Gorenstein on the pointed spectrum: by Corollary 4.1 Z is empty if and only if $\overline{P}$ is graded. $\overline{P}$ is a bounded poset so it is graded if and only if it is pure (see prop. 4.3 below) which holds if and only if P is pure.

Similarly, by Corollary 4.1 Z is zero-dimensional if and only if $P\dot{\cup}\{\infty\}$ and $P\dot{\cup}\{-\infty\}$ are graded. It is easy to show that this is equivalent to every connected component of P being pure. (for a proof consider Lemma 5.2 in [HHS19]).

The partially ordered set from example 4.2 defines a variety that is Gorenstein on the pointed spectrum but not Gorenstein since every component of P is pure while P is not.

4.4 Comparison to [MP20]

In the recently uploaded paper [MP20] on the arxiv, the Erhart rings of order-polytopes and of chain-polytopes associated to finite posets are investigated. Erhart rings of order-polytopes are exactly Hibi rings. The focus of the paper is to study the non-Gorenstein loci which is done by characterizing the generators of the trace ideal $\operatorname{tr}(\omega)$ combinatorially. In particular, for Hibi rings a family of graded primes is described in Theorem 4.5 (stated below) and the beginning of Chapter 5 that intersect in the radical $\sqrt{\operatorname{tr}(\omega)}$. We give a simpler proof of Theorem 4.5 using our approach and compare results.

Proposition 4.3. *Let A be a finite partially ordered set which is bounded and graded. Then A is pure.*

Proof. Let a be the minimal and b be the maximal element of A. Let ν denote a grading of A and let

$$a = a_1 \lessdot \cdots \lessdot a_r = b$$

be a maximal chain. It holds

$$r - 1 = \nu(a) - \nu(b),$$

showing that all maximal chains have the same length. □

Definition 4.12. Let A be a finite partially ordered set and $a \leq b \in A$. The rank $\mathrm{rank}(a, b)$ is defined to be the maximal length $r - 1$ of a chain

$$a = a_1 < a_2 < \cdots < a_r = b.$$

Similarly, the distance $\mathrm{dist}(a, b)$ is defined to be the minimal length $r - 1$ of an inclusion maximal chain

$$a = a_1 \lessdot a_2 \lessdot \cdots \lessdot a_r = b.$$

Remark 4.6. Every interval $[a, b]$ with grading $\nu : [a, b] \longrightarrow \mathbb{Z}$ is pure by Proposition 4.3. It holds

$$\mathrm{dist}(a, b) = \mathrm{rank}(a, b) = \nu(a) - \nu(b),$$

since every chain $a = a_1 \lessdot a_2 \lessdot \cdots \lessdot a_r = b$ is maximal.

Definition 4.13. Let u be a natural number and

$$a_1 < b_1 > a_2 < \cdots > a_u < b_u > a_1$$

be elements of $\overline{P}$ satisfying the inequality

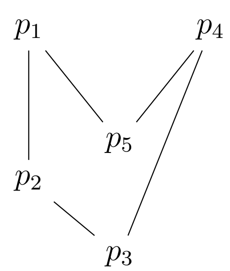
$$\sum_{i=1}^{u} \mathrm{rank}(a_i, b_i) > \mathrm{dist}(a_2, b_1) + \cdots + \mathrm{dist}(a_u, b_{u-1}) + \mathrm{dist}(a_1, b_u). \tag{4.1}$$

We define the graded prime ideal

$$\mathfrak{p}_{(a_1,\ldots,a_u,b_1,\ldots,b_u)} := \langle \chi^{\psi} | \psi \in C(P),\ \psi \text{ nonconstant on } \{a_1, \ldots, a_u, b_1, \ldots, b_u\} \rangle.$$

Example 4.3. All intervals contained in the following partially ordered set A

$A :$



are graded. The sequence

$$p_5 < p_4 > p_3 < p_1 > p_5$$

satisfies inequality (4.1). As can be seen by applying Lemma 4.2 below, this is a certificate that A itself is not graded.

Theorem 4.5 of [MP20] states

Theorem (4.5, [MP20])**.**

$$\sqrt{tr(\omega)} = \bigcap_{(a_1,\dots,a_u,b_1,\dots,b_u)} \mathfrak{p}_{(a_1,\dots,a_u,b_1,\dots,b_u)}$$

where u runs over all natural numbers and $(a_1, \dots, a_u, b_1, \dots, b_u)$ runs over all elements of $\overline{P}$ satistfying inequality (4.1)*.*

The idea is that elements $(a_1, \dots, a_u, b_1, \dots, b_u)$ satistying inequality (4.1) provide a certificate for the non-gradedness of subsets of $\overline{P}$ containing these elements. The monomials χ^{ψ} lying in the radical of $\mathrm{tr}(\omega)$ are characterized by ψ being nonconstant on all non-graded complete subsets of $\overline{P}$ (remark 4.4) and we prove that this is the case if and only if ψ is nonconstant on all elements $(a_1, \dots, a_u, b_1, \dots, b_u)$ satistying inequality (4.1).

Lemma 4.2. *A finite partially ordered set A is graded if and only if for every sequence of elements*

$$a_1 < b_1 > a_2 < \dots > a_u < b_u > a_1$$

in A the equation

$$\sum_{i=1}^{u} \mathrm{rank}(a_i, b_i) = \mathrm{dist}(a_2, b_1) + \dots + \mathrm{dist}(a_u, b_{u-1}) + \mathrm{dist}(a_1, b_u) \tag{4.2}$$

holds. Furthermore, if A is not graded there are elements

$$a_1 < b_1 > a_2 < \dots > a_u < b_u > a_1$$

satisfying inequality (4.1)*.*

Proof. For the first implication let $\nu : A \longrightarrow \mathbb{Z}$ be a grading and let

$$a_1 < b_1 > a_2 < \dots > a_u < b_u > a_1$$

be elements in A. For every relation $a_i < b_j$ it holds

$$\mathrm{rank}(a_i, b_j) = \mathrm{dist}(a_i, b_j) = \nu(a_i) - \nu(b_j)$$

by remark 4.6. This implies equation (4.2).

For the other direction we assume equality (5.1) always holds and we need to construct a grading of A. This is done by induction on the cardinality $\#A$: let $m \in A$ be a maximal element and $A' := A \setminus \{m\}$. By induction hypothesis there is a grading ν of A'. In order to extend it to A, ν needs to take the same value on all elements $x, y \in A$ with $x \lessdot m, y \lessdot m$. Since on each connected component of A', we can shift ν by integer values, we may assume A' to be connected. Hence there is a sequence

$$x = a_1 < b_1 > \cdots < b_{u-1} > a_u = y$$

of elements in A'. We set $b_u = m$ and equation (4.2) reads

$$\begin{aligned} &\operatorname{rank}(a_1, b_1) + \cdots + \operatorname{rank}(a_{u-1}, b_{u-1}) + \operatorname{rank}(a_u, m) \\ =&\operatorname{dist}(a_2, b_1) + \cdots + \operatorname{dist}(a_u, b_{u-1}) + \operatorname{dist}(a_1, m) \\ \Longrightarrow& \nu(a_1) - \nu(b_1) + \cdots + \nu(a_{u-1}) - \nu(b_{u-1}) + 1 \\ =&\nu(a_2) - \nu(b_1) + \cdots + \nu(a_u) - \nu(b_{u-1}) + 1 \\ \Longrightarrow& \nu(a_1) = \nu(a_u) \end{aligned}$$

by remark 4.6. This finishes the proof of the equivalence.

We now prove the last part of the lemma: let A be non-graded. If there is a non-graded interval $[a, b] \subseteq A$ then it is not pure by Proposition 4.3 and hence $\operatorname{rk}(a, b) > \operatorname{dist}(a, b)$. The choice $u = 1,\ a_1 = a,\ b_1 = b$ satisfies inequality (4.1). We now assume that all intervals in A are graded. Since A on the other hand is not, by the above results there exists a sequence

$$a_1 < b_1 > a_2 < \cdots > a_u < b_u > a_1$$

with

$$\sum_{i=1}^{u} \operatorname{rank}(a_i, b_i) \neq \operatorname{dist}(a_2, b_1) + \cdots + \operatorname{dist}(a_u, b_{u-1}) + \operatorname{dist}(a_1, b_u).$$

If the left side of the inequality is bigger then the proof is finished. Otherwise, since all intervals of A are graded, by remark 4.6 it holds $\operatorname{rank}(a_i, b_j) = \operatorname{dist}(a_i, b_j)$ for all $1 \leq i, j \leq u$. Defining $a'_1 = a_2, \ldots, a'_{u-1} = a_u, a'_u = a_1$ finishes the proof. $\square$

Theorem 4.5 of [MP20] now is a direct consequence of our results in Chapter 3:

proof of Theorem 4.5. $\sqrt{\operatorname{tr}(\omega)}$ is the intersection of all graded primes lying over $\operatorname{tr}(\omega)$ which we characterized in terms of surjective, order preserving maps $\phi : \overline{P} \longrightarrow P'$. As noted in remark 4.4 a monomial χ^ψ lies in $\sqrt{\operatorname{tr}(\omega)}$ if and only if ψ is nonconstant on every non-graded, complete subset $A \subseteq \overline{P}$. To show

$$\sqrt{\operatorname{tr}(\omega)} \subseteq \bigcap_{(a_1,\ldots,a_u,b_1,\ldots,b_u)} \mathfrak{p}_{(a_1,\ldots,a_u,b_1,\ldots,b_u)}$$

let $\chi^\psi \in \sqrt{\operatorname{tr}(\omega)}$ and $a_1, \ldots, a_u, b_1, \ldots, b_u$ be elements of $\overline{P}$ sytisfying inequality (4.1). We define

$$A := \bigcup_{i,j} [a_i, b_j]$$

to be the smallest complete set containing $\{a_1, \dots, a_u, b_1, \dots, b_u\}$. As shown in Lemma 4.2 A is non-graded. Since ψ is nonconstant on A, and ψ is order reversing, ψ is nonconstant on $\{a_1, \dots, a_u, b_1, \dots, b_u\}$ as well, showing the containment

$$\chi^\psi \in \mathfrak{p}_{(a_1,\dots,a_u,b_1,\dots,b_u)}.$$

For the other inclusion let

$$\chi^\psi \in \bigcap_{(a_1,\dots,a_u,b_1,\dots,b_u)} \mathfrak{p}_{(a_1,\dots,a_u,b_1,\dots,b_u)}$$

be an arbitrary homogeneous generator. Let A be a non-graded and complete subset of $\overline{P}$. By Lemma 4.2 A contains elements $a_1, \dots, a_u, b_1, \dots, b_u$ satisfying inequality (4.1) as in definition 4.13. The map ψ is nonconstant on $\{a_1, \dots, a_u, b_1, \dots, b_u\}$ and hence nonconstant on A as well showing

$$\chi^\psi \in \sqrt{\operatorname{tr}(\omega)}.$$

□

5 Secants of Segre varieties

Secants of Segre varieties are objects in classical algebraic geometry. They are connected to the border rank of tensors and to the computational complexity of matrix multiplication. In [MOZ15] the first secant variety is studied using methods from statistics. On affine patches, so called secant-cumulant coordinates are introduced which are a special case of the L-cumulants introduced in [Zwi12]. The change to these coordinates identifies open affine patches of the secant variety with affine, normal toric varieties. This covering allows us to apply our results from Chapter 3.

We start by introducing the notation and objects from [MOZ15]: let $k_1 \leq \cdots \leq k_n$ be natural numbers and consider the Segre map

$$\begin{aligned} \mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_n} &\longrightarrow \mathbb{P}^{(k_1+1)\cdots(k_m+1)-1} \\ ([a_0^1 : \cdots : a_{k_1}^1], \dots, [a_0^n : \cdots : a_{k_n}^n]) &\longmapsto \left[x_{i_1,\dots,i_n} = \prod_{j=1}^n a_{i_j}^j \right]_{\substack{0 \leq i_1 \leq k_1 \\ \vdots \\ 0 \leq i_n \leq k_n}}. \end{aligned}$$

The image of this map cinsists of all order one tensors and is called Segre variety. Its intersection with the affine open

$$U = \{x_{0,\dots,0} \neq 0\} \subseteq \mathbb{P}^{(k_1+1)\cdots(k_m+1)-1}$$

has a similar parametrization which we obtain by setting the variables $a_0^1, \ldots, a_0^n$ to 1:

$$\begin{aligned} \mathbb{A}^{k_1} \times \cdots \times \mathbb{A}^{k_n} &\longrightarrow U \\ ((a_1^1, \ldots, a_{k_1}^1), \ldots, (a_1^n, \ldots, a_{k_n}^n)) &\longmapsto \left[x_{i_1,\ldots,i_n} = \prod_{\substack{1 \le j \le n, \\ i_j \ne 0}} a_{i_j}^j \right]_{\substack{0 \le i_1 \le k_1 \\ \vdots \\ 0 \le i_n \le k_n}} . \end{aligned}$$

Our object of interest is the first secant variety $\mathrm{Sec}(\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_n})$ which is defined to be the Zariski-closure of the union of lines in $\mathbb{P}^{(k_1+1)\cdots(k_m+1)-1}$ that intersect the Segre variety in two distinct points. It consists of all tensors of border rank 2. In [MOZ15] this variety is shown to have a covering with affine toric varieties. We sketch the approach and proceed by applying results from Chapter 3.

In [MOZ15] a parametrization of the affine open patch

$$V := \mathrm{Sec}(\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_n}) \cap U$$

is provided: V is the Zariski closure of the image of the map

$$\begin{aligned} \left(\mathbb{A}^{k_1+\cdots+k_n}\right)^{\times 2} \times \mathbb{A}^1 &\longrightarrow V \\ \left((a_j^i), (b_j^i), t\right) &\longmapsto \left[x_{i_0,\ldots,i_n} = (1-t) \prod_{\substack{1 \le j \le n, \\ i_j \ne 0}} a_{i_j}^j + t \prod_{\substack{1 \le j \le n, \\ i_j \ne 0}} b_{i_j}^j \right]_{\substack{0 \le i_1 \le k_1 \\ \vdots \\ 0 \le i_n \le k_n}} . \end{aligned}$$

Under the coordinate change

$$x_{(i_1,\ldots,i_n)} \mapsto z_{(i_1,\ldots,i_n)}$$

on U to the Secant-cumulant coordinates one obatins a monomial parametrization of V (see [MOZ15], Lemma 3.1). In turn, the coordinate ring of V has monomial generators. These generators correspond to the lattice points of a normal polytope: it is proven in Theorem 7.3 and Proposition 7.13 of [MOZ15] that in the new coordinates V is the product $\mathbb{A}^{k_1+\cdots+k_n} \times X$ where X is the normal affine toric variety $\mathrm{Spec}(k[C \cap \mathbb{Z}^{1+k_1+\cdots+k_n}])$ associated to the cone C over the polytope

$$Q = \Delta_{k_1} \times \cdots \times \Delta_{k_n} \cap \{q_j^i | \sum_{i,j} q_j^i \ge 2\}.$$

This normal (see remark 7.15 in [MOZ15]) polytope is the product of n simplices of respective dimensions $k_1, \ldots, k_n$ intersected with the halfspace $\{\sum_{i=1}^n \sum_{j=1}^{k_i} q_j^i \ge 2\}$. In coordinates:

$$\begin{aligned} Q = \{ \, & \left(q^1_1, \dots, q^1_{k_1}, \dots, q^n_1, \dots, q^n_{k_n}\right) \in \mathbb{R}^{k_1+\dots+k_n} \mid \\ & q^i_j \geq 0 \ \forall 1 \leq i \leq n, 1 \leq j \leq k_i, \\ & 1 - \sum_{j=1}^{k_i} q^i_j \geq 0 \text{ for } 1 \leq i \leq n, \\ & \sum_{i=1}^{n} \sum_{j=1}^{k_i} q^i_j - 2 \geq 0 \} \end{aligned}$$

and

$$\begin{aligned} C = \{ \, & \left(q_0, q^1_1, \dots, q^1_{k_1}, \dots, q^n_1, \dots, q^n_{k_n}\right) \in \mathbb{R}^{1+(k_1+\dots+k_n)} \mid \\ & q^i_j \geq 0 \ \forall 1 \leq i \leq n, 1 \leq j \leq k_i, \\ & q_0 - \sum_{j=1}^{k_i} q^i_j \geq 0 \text{ for } 1 \leq i \leq n, \\ & \sum_{i=1}^{n} \sum_{j=1}^{k_i} q^i_j - 2q_0 \geq 0 \}. \end{aligned}$$

We now investigate which faces F of C contribute to non-Gorenstein locus Z of X by applying Lemma 3.3 and start with some observations about C and $C^\vee$. By construction of C, $C^\vee$ is the cone generated by the vectors

- $R^i_j := e^i_j, \quad \forall 1 \leq i \leq n, 1 \leq j \leq k_i,$
- $L_i := e_0 - \sum_{j=1}^{k_i} e^i_j, \quad \forall 1 \leq i \leq n,$
- $S := \sum_{i=1}^{n} \sum_{j=1}^{k_i} e^i_j - 2e_0.$

Here the vectors e^i_j denote the standard basis of $\mathbb{R}^{k_1+\dots+k_n}$. A list of the primitive ray generators of $C^\vee$ is given in the proof of Theorem 7.18 in [MOZ15]: in the case $n = 4$, or $n = 3$ and $k_1 > 1$ all generators of $C^\vee$ are ray generators. If $n = 3$ and $k_1 = 1$ all R^i_1 with $k_i = 1$ are omitted. In both cases $C^\vee$ is a pointed cone. For $n = 2$ on the other hand the lineality space of $C^\vee$ is spanned by S, L_1, L_2 and the ray generators of $C^\vee / \langle S, L_1, L_2 \rangle$ are the R^i_j.

Theorem 5.1. *If X is not Gorenstein, the non-Gorenstein locus Z of X is of dimension*

- $\max\{(k_{i'} - 1)(k_{i''} - 1) + 1$ *where* $1 \leq i' \neq i'' \leq n, \sum_{i \neq i', i''} k_i \neq 3\}$ *if* $n = 4$ *or* $n = 3, k_1 > 1,$
- $(k_2 - 1)(k_3 - 1) + 1$ *if* $n = 3, k_2 \neq 1,$
- 0 *if* $n = 3, k_1 = k_2 = 1,$
- 0 *if* $n = 2.$

Let $F \subseteq C$ be a face of positive dimension. We will investigate when F describes a maximal component of the non-Gorenstein locus by applying Lemma 3.3. The proof the theorem is split into three cases:

1. $n \geq 4$ or $n = 3$ and $k_1 \geq 2$,
2. $n = 3$ and $k_1 = 1$,
3. $n = 2$.

In the first two cases $C^\vee$ is pointed and we may apply Lemma 3.3 to the choice $\sigma = C^\vee$ and $\sigma^\vee = C$. In the last case the lineality space of $C^\vee$ needs to be taken care of and we choose $\sigma = C^\vee/\langle S, L_1, L_2\rangle$ and $\sigma^\vee = C \cap \langle S, L_1, L_2\rangle^\perp$.

Before going to the proof we need a lemma:

Lemma 5.1. *Let $n = 4$ or $n = 3$ and $k_1 > 1$. Then a positive-dimensional face $F \subseteq S^\perp$ contributes to the non-Gorenstein locus if and only if there are exactly two indices i', i'' with $F \subseteq L_{i'}^\perp$, $F \subseteq L_{i''}^\perp$ and $\sum_{i \notin \{i',i''\}} k_i \neq 3$ and in addition $F \subseteq S^\perp$.*

Proof. Since F is of positive dimension the intersection $F \cap Q$ with the polytope Q is nonempty. Observe that for each $1 \leq i \leq n$ the intersection of all facets of the k_i-dimensional simplex

$$Q \cap \bigcap_{j=1}^{k_i} (R_j^i)^\perp \cap F_i^\perp$$

is empty. Consequently F is not contained in every such facet: there is an index $1 \leq j \leq k_i$ with $F \nsubseteq (R_j^i)^\perp$ or $F \nsubseteq L_i^\perp$. We assume without loss of generality that F is not contained in the first hyperplane $(R_1^i)^\perp$: for each i with $F \subseteq L_i^\perp$ it holds $F \nsubseteq (R_1^i)^\perp$.

Assume $F \nsubseteq S^\perp$. Then we construct an integral element of $F[1]$: set $q_0 = k_n + 1$ and all $q_j^i = 1$ except that we set $q_1^i = k_n - k_i$ if $F \subseteq L_i^\perp$. All equations

$$q_j^i = 1 \ \forall 1 \leq i \leq n, 1 \leq j \leq k_i |\ F \subseteq (R_j^i)^\perp,$$
$$q_0 - \sum_{j=1}^{k_i} q_j^i = 1 \ \forall 1 \leq i \leq n |\ F \subseteq L_i^\perp$$

are satisfied. This shows that F does not contribute to the non-Gorenstein locus by Lemma 3.3 and from now on we assume $F \subseteq S^\perp$.

If $1 \leq i' \leq n$ is the only number with $F \subseteq L_{i'}^\perp$ then we can again construct an integral element of $F[1]$: the choices

$$q_0 = k_1 + \cdots + k_n - 5$$

and

$$q_j^i = 1$$

for all $1 \leq i \leq n, 1 \leq j \leq k_i$ except

$$q_1^{i'} = k_1 + \cdots + k_n - k_{i'} - 4$$

satisfy the equations

$$q_j^i = 1 \text{ if } F \subseteq (R_j^i)^\perp,$$
$$q_0 - \sum_{j=1}^{k_{i'}} q_j^{i'} = 1,$$
$$\sum_{i=1}^{n}\sum_{j=1}^{k_i} q_j^i - 2q_0 = 1.$$

Consequently, for F to contribute to the non-Gorenstein locus there need to be at least two distinct indices $1 \leq i', i'' \leq n$ with $F \subseteq L_{i'}^\perp, L_{i''}^\perp$. Assume we are in this case. By adding the three equations

$$q_0 - \sum_{j=1}^{k_{i'}} q_j^{i'} = 0, \;\; q_0 - \sum_{j=1}^{k_{i''}} q_j^{i''} = 0,$$
$$\sum_{i=1}^{n}\sum_{j=1}^{k_i} q_j^i - 2q_0 = 0$$

we obtain
$$\sum_{i \neq i', i''}\sum_{j=1}^{k_i} q_j^i = 0. \tag{5.1}$$

An additional equation $q_0 - \sum_{j=1}^{k_{i'''}} q_j^{i'''} = 0$ would leave us with only trivial solutions, so there are exactly two indices i', i'' with $F \subseteq L_{i'}^\perp, L_{i''}^\perp$.

Every integral element in $F[1]$ has to satisfy the equations

$$q_0 - \sum_{j=1}^{k_{i'}} q_j^{i'} = 1, \;\; q_0 - \sum_{j=1}^{k_{i''}} q_j^{i''} = 1,$$
$$\sum_{i=1}^{n}\sum_{j=1}^{k_i} q_j^i - 2q_0 = 1.$$

adding these equations leaves us with

$$3 = \sum_{i \neq i', i''}\sum_{j=1}^{k_i} q_j^i = \sum_{i \neq i', i''} k_i,$$

so if $\sum_{i \notin \{i', i''\}} k_i \neq 3$, F contributes to the non-Gorenstein locus. Conversely, if $\sum_{i \notin \{i', i''\}} k_i = 3$ the choice $q_0 = k_{i'} + k_{i''}$ and $q_j^i = 1$ for all i and j except $q_1^{i'} = k_{i''}$, $q_1^{i''} = k_{i'}$ defines an integral element of $F[1]$. $\square$

proof of theorem 5.1. In [MOZ15] (Theorem 7.18) it is shown that X is Gorenstein exactly in the following cases:

- $n = 5$ and $k_5 = 1$,
- $n = 3$ and $(k_1, k_2, k_3) \in \{(1,1,1),(1,1,3),(1,3,3),(3,3,3)\}$,
- $n = 2$ and $k_1 \in \{k_2, 1\}$.

Assume that we are not in any of these cases. We now determine the maximal faces F of C that contribute to the non-Gorenstein locus.

Case 1 $n = 4$ or $n = 3$ and $k_1 > 1$.
By the above lemma, the maximal faces contributing to the non-Gorenstein locus are of the form

$$F = C \cap L_{i'}^{\perp} \cap L_{i''}^{\perp} \cap S^{\perp}.$$

F is the cone over the product of the $k_{i'} - 1$ simplex and the $k_{i''} - 1$ simplex:

$$\begin{aligned} F = \{(q_0, q_1^{i'}, \dots, q_{k_{i'}}^{i'}, q_1^{i''}, \dots, q_{k_{i''}}^{i''})| \\ q_1^{i'} + \dots + q_{k_{i'}}^{i'} = q_0, \\ q_1^{i''} + \dots + q_{k_{i''}}^{i''} = q_0\} \end{aligned}$$

and of dimension $(k_{i'} - 1)(k_{i''} - 1) + 1$.

Case 2 $n = 3$ and $k_1 = 1$.
Case 2 is the same as case 1 except that all R_1^i with $k_i = 1$ are no ray generators of $C^{\vee}$. Consequently the faces

$$F = C \cap L_{i'}^{\perp} \cap L_{i''}^{\perp} \cap S^{\perp}$$

where $1 \in \{k_{i'}, k_{i''}\}$ do not contribute to Z. Reiterating the proof of case 1 leaves us with the desired result.

Case 3 $n = 2$.
C is a fulldimensional subcone of the linear space $S^{\perp} \cap L_1^{\perp} \cap L_2^{\perp}$ so we may apply Lemma 3.3. As we have observed in case 1 since F is of positive dimension, for both indices $i = 1$ and $i = 2$ there is a hyperplane $(R_j^i)^{\perp}$ not containing F. We may assume without loss of generality that $F \not\subseteq (R_1^1)^{\perp}, (R_1^2)^{\perp}$. The primitive ray generators of the dual cone $C^{\vee}/\langle L_1, L_2, S\rangle$ are exactly the vectors R_j^i. Hence an integral element $(q_j^i)_{i,j}$ of $F[1]$ is an integral element of $\langle L_1, L_2, S\rangle^{\perp}$ satisfying the equations

$$q_j^i = 1 \ \forall i \in \{1, 2\},\ j \neq 1.$$

The choice $q_1^1 = k_2, q_1^2 = k_1$, and $q_0 = k_1+k_2-1$ defines an integral element of $\langle L_1, L_2, S\rangle^\perp$ since the equations

$$q_0 - \sum_{j=1}^{k_1} q_j^1 = 0,\ q_0 - \sum_{j=1}^{k_2} q_j^2 = 0.$$
$$\sum_{i=1}^{2}\sum_{j=1}^{k_i} q_j^i - 2q_0 = 0$$

are satisfied. Consequently F does not contribute to the non-Gorenstein locus and Z is zero-dimensional. □

Corollary 5.1. *The dimension of the non-Gorenstein locus of* $\mathrm{Sec}(\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_n})$ *is the dimension of the non-Gorenstein locus of X plus $k_1 + \cdots + k_n$.*

Proof. We have seen that there is an affine covering of $\mathrm{Sec}(\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_n})$ with toric varieties isomorphic to $\mathbb{A}^{k_1+\cdots+k_n} \times X$. We now prove that the non-Gorenstein locus $\mathfrak{Z}$ of these patches is the product $\mathbb{A}^{k_1+\cdots+k_n} \times Z$ of affine space with the non-Gorenstein locus Z of X.

Observe that $\mathbb{A}^{k_1+\cdots+k_n} \times X$ is the toric variety associated to the product of cones $\mathbb{R}_{\geq 0}^{k_1+\cdots+k_n} \times C$. The torus-invariant subvarieties of $\mathbb{A}^{k_1+\cdots+k_n} \times X$ are products of torus-invariant subvarieties of $\mathbb{A}^{k_1+\cdots+k_n}$ and X respectively and analogously every face of the product cone is a product $F_1 \times F_2$ of a face F_1 of $\mathbb{R}_{\geq 0}^{k_1+\cdots+k_n}$ and a face F_2 of C. It suffices to show that $F_1 \times F_2$ contributes to the non-Gorenstein locus of $\mathbb{A}^{k_1+\cdots+k_n} \times X$ if and only if F_2 contributes to the non-Gorenstein locus of X. Since $F_1[1]$ contains the integral element $(1, \ldots, 1)$, the equality

$$(F_1 \times F_2)[1] = F_1[1] \times F_2[1]$$

implies that $(F_1 \times F_2)[1]$ contains an integral element if and only if $F_2[1]$ does, finishing the proof. □

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